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THE HYPERBOLIC METRIC AND THE GEOMETRY OF THE UNIVERSAL CURVE

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0. Introduction

0.1. A basic deformation of a compact Riemann surface is *pinching a nontrivial loop.* Understanding the limiting case, where the loop *degenerates to a point,* is actually tantamount to understanding the Deligne-Mumford stable curve compactification $\overline{\mathscr{M}_g}$ of the classical moduli space of Riemann surfaces. Degenerating families of Riemann surfaces are readily given by cut and paste constructions in the complex analytic category or by hyperbolic geometry, following Fenchel and Nielsen. A basic question is to relate the two approaches; to find the expansion for the hyperbolic metric in terms of the complex parametrization. The motivation is twofold, we would like to be able to analyze the degeneration of an invariant of hyperbolic metrics by simply writing out its expansion in complex coordinates, and we would like to use the hyperbolic metric in the study of the analytic geometry of $\overline{\mathscr{M}_g}$.

The following example will play a central role in our investigation. Consider in $\mathbb{C}^3$ the smooth germ of a variety $V = \{(z,w,t) | zw = t, |z|, |w|, |t| < 1\}$ with projection $\Pi\colon V \to D$, $\Pi((z,w,t)) = t$, to the unit disc. The projection is almost a fibration: the t_0, fiber $t_0 \neq 0$, is an annulus $\{|t_0| < |z| < 1,\ w = t_0/z\}$, while the 0-fiber is two transverse discs $\{|z| < 1\} + \{|w| < 1\}$, intersecting only at the origin. $V \to D$ is a degenerating family of annuli. Each fiber of $V_0 = V - \{\text{origin}\}$ has a complete hyperbolic metric: the t_0-fiber,

$$ds_{t_0}^2 = \left(\frac{\pi}{\log|t_0|} \csc \frac{\pi \log|z|}{\log|t_0|} \left|\frac{dz}{z}\right|\right)^2$$

and the 0-fiber,

$$ds_0^2 = \left(\frac{|d\zeta|}{|\zeta| \log|\zeta|}\right)^2, \qquad \zeta = z, w.$$

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