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INFINITESIMAL RIGIDITY FOR SMOOTH ACTIONS OF DISCRETE SUBGROUPS OF LIE GROUPS

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1. Introduction

Let G be a connected noncompact semisimple Lie group with finite center and no compact factors, and let $\Gamma \subset G$ be a cocompact discrete subgroup. If G has no simple factor locally isomorphic to any $SO(1,n)$ or $SU(1,n)$, we established in [15], [18], [20] a rigidity property for any isometric action of Γ on a compact Riemannian manifold M. Namely, we proved that any sufficiently small perturbation of an isometric action (in the topology of pointwise convergence on Γ with the C^∞-topology on $\mathrm{Diff}(M)$) which is ergodic must also leave some smooth metric invariant. In this paper, we examine some rigidity properties of nonisometric actions of Γ. All presently known ergodic volume preserving actions of Γ on compact manifolds are of an algebraic nature (at least in the case in which G is of higher rank, i.e., all simple factors of G have real rank at least 2), and it is these algebraically defined actions which we examine. More precisely, let $\pi: G \to H$ be a homomorphism of G into another Lie group H, and let $\Lambda \subset H$ be a cocompact discrete subgroup. Then Γ acts on $M = H/\Lambda$ via π. Under mild assumptions this action will be ergodic (see [6, Chapter 2]). Unless π is trivial, there is no Γ-invariant Riemannian metric on H/Λ.

We recall that an action of a group Γ on a manifold M is called infinitesimally rigid if $H^1(\Gamma, \mathrm{Vect}(M)) = 0$, where $\mathrm{Vect}(M)$ is the space of smooth vector fields on M. This definition is of course motivated by the fact that $\mathrm{Vect}(M)$ is the Lie algebra of the infinite dimensional group $\mathrm{Diff}(M)$, and the natural action of Γ on $\mathrm{Vect}(M)$ is simply the composition of the action $\Gamma \to \mathrm{Diff}(M)$ with the adjoint representation of $\mathrm{Diff}(M)$ on its Lie algebra. We shall call an action L^2-infinitesimally rigid if the map $H^1(\Gamma, \mathrm{Vect}(M)) \to H^1(\Gamma, \mathrm{Vect}_2(M))$ is zero, where $\mathrm{Vect}_2(M)$ is the space of L^2-vector fields on M. We can then state our main results as follows.

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