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CONVERGENCE OF THE RICCI FLOW FOR METRICS WITH INDEFINITE RICCI CURVATURE

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Abstract

Hamilton's Ricci flow convergence theorems generally deal with metrics whose Ricci curvature is positive semidefinite. Here, we exhibit a nontrivial class of three-dimensional Riemannian metrics with Ricci curvature of indefinite sign for which the Ricci flow converges.

Recent work of Hamilton [5], [6] shows that for all 3-dimensional Riemannian geometries (Σ^3, g) with positive Ricci curvature, the "Ricci flow" generated by the (heat-like) equation

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} + \frac{2}{3} g_{ij} r \tag{1}$$

(for $r := \int_{\Sigma^3} R\, d\mu / \int_{\Sigma^3} d\mu$, the average of the scalar curvature R over Σ^3) converges asymptotically in parameter time t to a metric of constant positive curvature. The nature of the proof of this result has led to speculation that for Riemannian geometries with Ricci curvature of indefinite sign, the Ricci flow would generally not converge. The product geometry $S^2 \times S^1$, whose Ricci flow approaches a singular "pinching" geometry, provides an example of the expected behavior.

Here, we consider a class of 3-dimensional geometries, all with nondefinite or negative Ricci curvature, for which the Ricci flow does indeed always converge. This class, which we shall label $\mathscr{F}$, is fairly specialized. All its members are topologically T^3 (3-torus) and all are invariant under a freely acting T^2 isometry group. (There are a few other orthogonality conditions which characterize $\mathscr{F}$; we spell them out in §1.) The class is nontrivial, however, and as we shall prove in §2, all metrics in $\mathscr{F}$ converge (under Ricci flow) to a flat metric on T^3.

The convergence proof, though fairly intricate in detail, has a single theme: showing that the scalar curvature R decays sufficiently rapidly as

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