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EINSTEIN METRICS ON PRINCIPAL TORUS BUNDLES

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0. Introduction

The goal of this paper is to describe a new class of Einstein metrics and discuss their geometrical and topological properties.

The building blocks for these examples are Kähler-Einstein manifolds with positive first Chern class. Let (M, g) be such a manifold. Then there exists a positive integer q such that $c_1(M) = q\alpha$ and α is an indivisible class in $H^2(M; \mathbb{Z})$.

Theorem 1. *Let (M_i, g_i), $i = 1, \cdots, m$, be Kähler-Einstein manifolds with $c_1(M_i) > 0$ and $c_1(M_i) = q_i\alpha_i$ with α_i indivisible. Let P be the total space of a principal torus bundle over $B = M_1 \times \cdots \times M_m$ whose characteristic classes in $H^2(B; \mathbb{Z})$ are integral linear combinations of $\alpha_1, \cdots, \alpha_m$. Then P carries an Einstein metric with positive scalar curvature iff $\pi_1(P)$ is finite.*

The condition that $\pi_1(P)$ be finite is necessary for the existence of an Einstein metric with positive scalar curvature by the theorem of Bonnet and Myers. In the special case of circle bundles we get

Corollary 1. *Every nontrivial circle bundle over $M_1 \times \cdots \times M_m$ whose Euler class is an integral linear combination of $\alpha_1, \cdots, \alpha_m$ carries an Einstein metric with positive scalar curvature.*

In the case of these circle bundles we also show that the Einstein metric we obtained is uniquely determined up to scaling by the property that the projection $P \to M_1 \times \cdots \times M_m$ is a Riemannian submersion with totally geodesic fibers and that the metric on the base is a product of Kähler-Einstein metrics. In general, the metric on the base will not be Einstein.

In the special cases of circle bundles over $P^1\mathbb{C} \times P^2\mathbb{C}$ and over $P^1\mathbb{C} \times P^1\mathbb{C} \times P^1\mathbb{C}$, these Einstein metrics were discovered independently by the physicists R. D'Auria, Castellani, Fré, and van Nieuwenhuizen [3], [7] in

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