J. DIFFERENTIAL GEOMETRY
31 (1990) 185-213

THE ADIABATIC LIMIT, HODGE COHOMOLOGY AND LERAY'S SPECTRAL SEQUENCE FOR A FIBRATION

RAFE R. MAZZEO & RICHARD B. MELROSE

0. Introduction

Consider a fibration of a compact manifold, M, with fibers modeled differentially on the compact manifold F:

$$\begin{array}{ccc} F & \longrightarrow & M \\ & & \downarrow{\scriptstyle \pi} \\ & & Y \end{array}$$

Here, and throughout, we work with $\mathscr{C}^\infty$ maps and spaces. Let h be a Riemannian metric on the base space Y. Suppose that $g_\infty \in \mathscr{C}^\infty(M; S^2M)$ is a $\mathscr{C}^\infty$ symmetric 2-cotensor on M which is positive definite on the fibers. In particular g_∞ induces a Riemannian structure on each fiber. Combining these two forms gives a 1-parameter family of metrics on M:

$$g_x = g_\infty + x^{-2}\pi^* h \qquad (x > 0). \tag{1}$$

Let Δ_x be the Laplace-Beltrami operator of this metric and let $\mathscr{H}_x$ be the null space of Δ_x, the space of g_x-harmonic forms on M. We are interested in the behavior of $\mathscr{H}_x$ as $x \downarrow 0$, the so-called 'adiabatic limit', as discussed recently—with somewhat different objectives—by Bismut [1], Bismut and Freed [3], Cheeger [5] and Bismut and Cheeger [2] (see also Witten [11]). We shall show that $\mathscr{H}_x$ has a basis which extends to be smooth for $x \in [0, \infty)$. This basis remains independent at $x = 0$, spanning $\mathscr{H}_0 = H^k_{\mathrm{HL}}(M)$ (which we call the Hodge-Leray cohomology), and this limiting space represents the cohomology of M. The determination of which forms on M lie in the limiting space $H^k_{\mathrm{HL}}(M)$ can be carried out by a Taylor series analysis. We show that this approach gives a Hodge theoretic version of Leray's spectral sequence for the cohomology of M. An attractive treatment of the spectral sequence can be found in [4].

Received January 29, 1988 and, in revised form, August 11, 1988. This research was supported in part by the National Science Foundation under a Postdoctoral Fellowship and Grant DMS-8603523.