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MUTATION AND THE η-INVARIANT

ROBERT MEYERHOFF & DANIEL RUBERMAN

1. Introduction

This paper investigates the effect of a certain type of cutting and pasting operation on geometric invariants of a hyperbolic 3-manifold. The invariants which we discuss are the Chern-Simons invariant [4] and the η-invariant [2]. These are both defined for any closed Riemannian 3-manifold M; the Chern-Simons invariant $\mathrm{CS}(M)$ takes its values in the circle $\mathbf{R}/\mathbf{Z}$, while the η-invariant $\eta(M)$ is a real number. The two numbers are related as follows: $3\,\mathrm{CS}(M) = 2\eta(M) \pmod 1$. Hence the η-invariant determines the Chern-Simons invariant, modulo $1/2$. If M is a compact hyperbolic manifold, then Mostow's rigidity theorem [9] implies that both of these are in fact topological invariants of M. The Chern-Simons invariant is defined, modulo $1/2$, for finite-volume hyperbolic manifolds in [8]. The extension of Mostow rigidity to the finite volume case [11] implies that the Chern-Simons invariant is a topological invariant in this case as well.

If F is a surface embedded in a 3-manifold M, and φ is a diffeomorphism of F, then we can obtain a new manifold M^φ by cutting M along F and regluing via φ. For an arbitrary Riemannian manifold, this topological operation is likely to destroy any nice properties of the metric. In particular, there is no reason to expect any relation between invariants of the two manifolds. However, in certain special cases, the cutting and pasting may be done "geometrically." Let F be a surface of genus two and τ be the involution of F indicated in Figure 1. If F is embedded in a hyperbolic 3-manifold, then cutting and pasting via τ may be done geometrically:

Theorem 1.1 (*Ruberman* [12]). *Suppose the genus-2 surface F is embedded in the hyperbolic manifold M so that $\pi_1(F)$ injects into $\pi_1(M)$. Then the manifold M^τ is hyperbolic, and* $\mathrm{vol}(M^\tau) = \mathrm{vol}(M)$.

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