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THE LICHNEROWICZ CONJECTURE ON HARMONIC MANIFOLDS

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0. Introduction

The theory of harmonic manifolds has a relatively long history. It started with a work of H. S. Ruse in 1930, who made an attempt to find a solution for the equation $\Delta f = 0$ on a general Riemannian manifold which depends only on the geodesics distance $r(x, \cdot)$. His main aim was to use these functions to develop harmonic analysis on Riemannian manifolds similar to the euclidean case.

It turned out that such radial harmonic functions exist only in very special cases, namely, in the cases where the density function $\omega_p := \sqrt{|\det g_{ij}|}$ in the normal coordinate neighborhood $\{x^1, \cdots, x^n\}_p$ around each point p depends only on $r(p, \cdot)$. From the well-known symmetry $\omega_p(q) = \omega_q(p)$ it can be easily seen that this is the case if and only if the function $\omega_p(q)$ is of the form

$$\omega_p(q) = \phi(r(p,q)); \qquad \phi\colon \mathbf{R}_+ \to \mathbf{R},$$

where $\mathbf{R}_+$ is the set of all positive real numbers, and $\mathbf{R}$ is the set of all real numbers. A Riemannian manifold was defined to be harmonic precisely when its density function $\omega_p(q)$ satisfies this radial property.

For a precise formulation one can introduce the notions of *global*, *local*, and *infinitesimal harmonicity* [5]. The global (respectively, local) harmonicity refers to the case where the above radial property of the density function is global (respectively, local). For infinitesimal harmonicity we assume only that the derivatives $\nabla^{(k)}_{\xi_p \cdots \xi_p} \omega_p$ with respect to the unit vectors $\xi_p \in T_p(M^n)$ define constant functions on the manifold. These notions are obviously equivalent for analytic Riemannian manifolds [5].

The derivatives $\nabla^{(k)}_{\xi_p \cdots \xi_p} \omega_p$ can be expressed with the help of the curvature tensor and its covariant derivatives. For example, we have

$$\nabla^{(2)}_{\xi_p \xi_p} \omega_p = -\frac{1}{3} R(\xi_p, \xi_p),$$

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