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WITTEN'S COMPLEX AND INFINITE DIMENSIONAL MORSE THEORY

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Abstract

We investigate the relation between the trajectories of a finite dimensional gradient flow connecting two critical points and the cohomology of the surrounding space. The results are applied to an infinite dimensional problem involving the symplectic action function.

1. Introduction

Let M be a smooth finite dimensional manifold and let $f\colon M \to \mathbb{R}$ be a smooth function. It is the aim of Morse theory to relate the topological type of M and the number and types of critical points of f, i.e. of points $x \in M$ with $df(x) = 0$. For example, if M is compact and all critical points of f are nondegenerate, then there are the well-known Morse inequalities (see e.g. [6]) relating the number of critical points and their Morse indices to the dimension of the graded vector spaces $H^*(M, \mathbb{F})$, where $\mathbb{F}$ is any field and $H^*(M, \mathbb{F})$ is the graded cohomology of M with coefficients in $\mathbb{F}$. (Throughout the paper, we will use Alexander-Spanier cohomology; see [12].) The Morse inequalities are usually stated as a relation between polynomials in $\mathbb{F}[t]$, but can be formulated equivalently as follows: Let us denote by $C^*_{\mathbb{F}}$ the free $\mathbb{F}$-vector space over the set $\mathbf{C}$ of critical points of f. That is, $\mathbf{C}^*_{\mathbb{F}} \simeq (\mathbb{F})^{|C|}$, is identified with a set of generators of $\mathbf{C}^*_{\mathbb{F}}$. Then the Morse inequalities are equivalent to the existence of an $\mathbb{F}$-linear map, called a coboundary operator $\delta_{\mathbb{F}} : \mathbf{C}^*_{\mathbb{F}} \to \mathbf{C}^*_{\mathbb{F}}$ so that $\delta_{\mathbb{F}}\delta_{\mathbb{F}} = 0$ and

$$\ker \delta_{\mathbb{F}} / \operatorname{im} \delta_{\mathbb{F}} = H^*(M, \mathbb{F}). \tag{1.1}$$

The central tool in the proof of the Morse inequalities is the gradient flow of f: If g is a Riemannian metric on M, and $\nabla_g f$ denotes the gradient vector field of f with respect to g, then the solutions of the ordinary differential equation

$$\dot{x}(t) + \nabla_g f(x(t)) = 0, \qquad x(0) = x, \tag{1.2}$$

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