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COMPLEX MANIFOLD GEOGRAPHY IN DIMENSION 2 AND 3

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Introduction

The term geography is used to describe the distribution of Chern numbers of algebraic manifolds of general type. The term was introduced by Persson [73], where he studied the question for algebraic surfaces of general type. Although the Chern numbers are quite rough invariants of manifolds, they are in general easily calculated, and are perhaps the only such invariants for algebraic manifolds of general type. Furthermore, there seem to be in general certain bounds on these Chern numbers, so the natural question arises as to whether there exists an algebraic manifold Y for every given set of numbers fulfilling the bounds, such that Y has precisely those Chern numbers. In the second part of this paper, which is mainly expository, we study these questions for algebraic 3-folds. To put this in its proper perspective, in the first part we review most of the constructions used to get the known results in the surface case. This subject has a history going back at least to about 1950.

About that time Thom created his cobordism theory [86] by which the set of cobordism classes of differentiable manifolds became a ring under the operations of cartesian product and disjoint sum, each class being characterized by its Pontrajagin numbers. The signature theorem, proven in 1953 [24], showed that the Pontrajagin numbers could not be arbitrary: there are integrality conditions which the Pontrajagin numbers of smooth manifolds had to fulfill. A complex analogue was developed by Milnor [61], and the Riemann-Roch theorem, proven in 1954 [24], showed that also the Chern numbers of (almost) complex manifolds had to fulfill certain integrality relations (which were classically known in (complex) dimensions 1 and 2). Milnor was then able to show that these integrality conditions are essentially *all* the conditions which must be fulfilled in the complex cobordism ring. Hirzebruch suggested then that if one restricts

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