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A TORELLI-TYPE THEOREM FOR GRAVITATIONAL INSTANTONS

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1. Introduction

In an earlier paper [10], a construction was described which produced families of 4-dimensional hyper-Kähler manifolds (one family for each finite subgroup of SU(2)), the members of which were asymptotically locally Euclidean (ALE). Our purpose here is to demonstrate the completeness of this construction: we shall show that every ALE hyper-Kähler 4-manifold is isometric to a member of one of the families obtained in [10].

For us, a Riemannian 4-manifold is ALE if it has just one end and if some neighborhood of infinity has a finite covering space $\tilde{V}$ diffeomorphic to the complement of the unit ball in $\mathbf{R}^4$; the Riemannian metric g^{ij} on $\tilde{V}$ is required asymptotically to approximate the Euclidean metric δ^{ij} on $\mathbf{R}^4$, so that in the natural coordinates x_i one has

$$g^{ij} = \delta^{ij} + a^{ij}$$

with $\partial^p a^{ij} = O(r^{-4-p})$, $p \geq 0$, where $r^2 = \sum x_i^2$ and ∂ denotes differentiation with respect to the coordinates x_i. We recall that a hyper-Kähler manifold carries three complex structures I, J, K and that these give three (closed) Kähler 2-forms $\omega_1, \omega_2, \omega_3$. With this notation, the main result of [10] is the following. Let Γ be a finite subgroup of SU(2) and let X be the smooth 4-manifold underlying the minimal resolution of the complex quotient singularity $\mathbf{C}^2/\Gamma$.

Theorem 1.1. *Let three cohomology classes $\alpha_1, \alpha_2, \alpha_3 \in H^2(X; \mathbf{R})$ be given which satisfy the nondegeneracy condition*

(*) *for each $\Sigma \in H_2(X; \mathbf{Z})$ with $\Sigma \cdot \Sigma = -2$ there exists $i \in \{1,2,3\}$ with $\alpha_i(\Sigma) \neq 0$.*

Then there exists on X an ALE hyper-Kähler structure for which the cohomology classes of the Kähler forms $[\omega_i]$ are the given α_i.

The results of this paper were announced in [10]. They comprise the following two theorems, which will be proved in §§2 and 3, respectively.

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