J. DIFFERENTIAL GEOMETRY
29 (1989) 665-683

THE CONSTRUCTION OF ALE SPACES AS HYPER-KÄHLER QUOTIENTS

P. B. KRONHEIMER

1. Introduction

According to the definition given by Calabi [4], a Riemannian manifold (X, g) is *hyper-Kähler* if it is equipped with three automorphisms I, J, K of the tangent bundle which satisfy the relations of the quaternion algebra $\mathbf{H}$ and are covariant constant with respect to the Levi-Civita connection:

$$I^2 = J^2 = K^2 = -1, \quad IJ = -JI = K, \quad \nabla I = \nabla J = \nabla K = 0.$$

These conditions imply in particular that each of I, J and K defines an integrable complex structure on X and that the metric g is Kähler with respect to all three; the three Kähler forms $\omega_1, \omega_2, \omega_3$ are therefore closed, giving three symplectic structures to X. In dimension 4, a simply-connected Riemannian manifold admits such a hyper-Kähler structure precisely when the Riemann curvature tensor is either self-dual or anti-self-dual. A complete, hyper-Kähler 4-manifold is therefore a self-dual, positive-definite solution to Einstein's equations in vacuum (a self-dual *gravitational instanton*), and it is with examples of such manifolds that we are concerned.

This paper describes the construction of a particular family of hyper-Kähler 4-manifolds, the so-called ALE spaces [6]. ALE stands for *asymptotically locally Euclidean* and describes a Riemannian 4-manifold with just one end which at infinity resembles a quotient $\mathbf{R}^4/\Gamma$ of Euclidean space $\mathbf{R}^4$ by a finite group Γ of identifications. The Riemannian metric g is required to approximate the Euclidean metric up to $O(r^{-4})$,

$$g^{ij} = \delta^{ij} + O(r^{-4}),$$

with appropriate decay in the derivatives of g^{ij}. A large class of such ALE spaces was discovered by Gibbons and Hawking [7]. For each integer $k \geq 2$, they constructed a family of spaces, depending on $3k - 6$ parameters, which had self-dual curvature and resembled at infinity a quotient of $\mathbf{R}^4$ by a cyclic group Γ of order k. These 'multi-Eguchi-Hanson' metrics were obtained also by Hitchin [8], who derived them by an application of Penrose's nonlinear

Received June 12, 1987 and, in revised form, January 29, 1988.