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THE TEICHMÜLLER THEORY OF HARMONIC MAPS

MICHAEL WOLF

1. Introduction

Let M be a smooth, closed, compact surface of genus $g \geq 2$, let $\mathscr{M}_{-1}$ denote the space of constant curvature -1 metrics on M, and let $\sigma|dz|^2$ denote a particular element of $\mathscr{M}_{-1}$. The group of diffeomorphisms isotopic to the identity, Diff_0, acts by pull back on $\mathscr{M}_{-1}$, and we can define the Teichmüller space of genus g, T_g, to be the quotient space $\mathscr{M}_{-1}/\mathrm{Diff}_0$. In studying Teichmüller space, it is natural to pick out a particular hyperbolic metric from each class; here we choose as our representatives the metrics $\rho|dw|^2$ which have the property that the map $\mathrm{id}\colon (M, \sigma|dz|^2) \to (M, \rho|dw|^2)$, which is the identity as a map on M, is harmonic as a map of Riemannian manifolds.

Naturally associated to a harmonic map $\mathrm{id}\colon (M, \sigma, z) \to (M, \rho, w)$ is a quadratic differential $\Phi(\sigma, \rho)\, dz^2$, which is holomorphic with respect to the conformal structure of σ. This then defines a map $\Phi(\sigma, \cdot)\colon T_g \to \mathrm{QD}(\sigma)$ from the Teichmüller space to the space of holomorphic quadratic differentials on (M, σ).

Sampson [17] showed that Φ is injective and continuous; here we first show (Theorem 3.1) that it is also surjective so that, via Φ^{-1}, $\mathrm{QD}(\sigma)$ provides global coordinates for T_g. The rest of this paper is an investigation of those coordinates.

Thurston ([5], [22]) introduced a compactification $\overline{T_g^{th}}$ of T_g that differed from the previous compactifications of T_g in that the action of the mapping class group (isotopy classes of orientation preserving diffeomorphisms) on T_g extended continuously to the boundary $\partial\overline{T_g^{th}}$. From the homeomorphism $\Phi\colon T_g \approx \mathrm{QD}(\sigma)$, we also obtain a compactification $\overline{T_g^h}(\sigma)$ of T_g given by adjoining points at ∞ to the rays of the vector space $\mathrm{QD}(\sigma)$. We show (Theorem 4.1) that $\overline{T_g^n(\sigma)}$ is the Thurston compactification $\overline{T_g^{th}}$, and so $\overline{T_g^h}(\sigma)$ is independent of the choice of σ as a base hyperbolic metric.

Naturally associated (see [10]) to a holomorphic quadratic differential Φdz^2 is a pair of measured foliations on M, topological objects. At all points $p \in M$

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