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EINSTEIN MANIFOLDS OF DIMENSION FIVE WITH SMALL FIRST EIGENVALUE OF THE DIRAC OPERATOR

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1. Introduction

Let M^n be a compact Einstein spin manifold with positive scalar curvature $R > 0$ and denote by $D\colon \Gamma(S) \to \Gamma(S)$ the Dirac operator acting on sections of the spinor bundle. If λ_1 is the first eigenvalue of this operator we have

$$\lambda_1^2 \geq \frac{1}{4}\frac{n \cdot R}{n-1}$$

(see e.g. [4]). Thus, there arises the interesting problem to classify all those Einstein spaces where the lower bound $\pm\frac{1}{2}\sqrt{\frac{(n\cdot R)}{(n-1)}}$ actually is an eigenvalue of the Dirac operator. The corresponding eigenspinor ψ satisfies the stronger equation

$$\nabla_X \psi = \mp\frac{1}{2}\sqrt{\frac{R}{n(n-1)}}X \cdot \psi$$

(see e.g. [4]) and these spinors are sometimes called Killing spinors (see e.g. [9], [16]). In case $n = 4$ the only possible manifold is $M^4 = S^4$ (see e.g. [5]).

In dimension six each solution of the equation $D\psi = \frac{1}{2}\sqrt{(6 \cdot R)/5}\psi$ defines a (nonintegrable) almost complex structure (see e.g. [8]). Furthermore, the assumption that $\pm\frac{1}{2}\sqrt{(n \cdot R)/(n-1)}$ is an eigenvalue of the Dirac operator imposes algebraic conditions on the Weyl tensor of the space (see e.g. [5]) as well as on the covariant derivative of the curvature tensor and the harmonic forms on M^n (see e.g. [9]). On the other hand, in the dimensions 5,6,7 examples of Einstein spaces different from the sphere are known for which $\pm\frac{1}{2}\sqrt{(n \cdot R)/(n-1)}$ is an eigenvalue of the Dirac operator (see e.g. [4], [7], [17]). Moreover, if M^n is a Kähler manifold, K.D. Kirchberg proved the stronger inequality $\lambda_1^2 \geq \frac{1}{4}(n+2)R/n$ (see e.g. [12]) and solved in the complex dimension $n/2 = 3$ the corresponding classification problem (see e.g. [13]); the only possible Einstein-Kähler spaces of complex dimension three realizing $\sqrt{\frac{2}{3}R}$ as an eigenvalue of the Dirac operator are $P^3(\mathbf{C})$ and $F(1,2)$

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