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STABILITY OF SINGULARITIES OF MINIMIZING HARMONIC MAPS

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1. Introduction

Singularities of energy minimizing harmonic maps may occur with 3-dimensional domains. Perhaps the simplest example is the map $x/|x|$ which has least energy [2] among all finite energy maps from the 3-ball $\mathbf{B}$ to the 2-sphere $\mathbf{S}^2$ having boundary values given by the identity map of $\mathbf{S}^2$. Moreover, in dimension 3, singularities are, by the work of R. Schoen and K. Uhlenbeck [10, Theorem], [11, 2.7], at most isolated. As the boundary data varies the singularities presumably move. In [5] was noted the impossibility of a sequence of minimizing configurations in which a pair of oppositely oriented singularities come together and cancel, leaving a singularity-free configuration. This followed from the strong convergence of minimizers and the basic small energy regularity theorem [10, 2.6]. These arguments left open the possibility of three singularities, two oppositely oriented, merging and leaving a single singularity. This is not precluded by either topological degree considerations or by the monotonicity of energy [11, 2.4]. However, the estimates of the present paper, in particular, rule out any such cancellation. Our results are based on the following:

Perturbation Lemma. *There exist positive constants δ_0, c_0, and α so that if $\varphi \in \operatorname{Lip}(\mathbf{S}^2, \mathbf{S}^2)$, $\delta = \|\varphi - \mathrm{id}_{\mathbf{S}^2}\|_{\mathrm{Lip}} \le \delta_0$, and $u \in H^1(\mathbf{B}, \mathbf{S}^2)$ is energy minimizing with $u|\mathbf{S}^2 \equiv \varphi$, then u has only one singular point a,*

$$|a| \le c_0 \delta^{1/2} \quad \text{and} \quad \left\| u - \theta\left(\frac{x-a}{|x-a|}\right)\right\|_{C^\alpha} \le c_0 \delta^{1/4}$$

for some orthogonal rotation θ of $\mathbf{R}^3$ with $\|\theta - \mathrm{id}_{\mathbf{R}^3}\| \le c_0 \delta^{1/4}$.

This leads to the following general

Stability Theorem. *Suppose Ω is a smooth bounded domain in $\mathbf{R}^3$, $\psi \in \operatorname{Lip}(\partial\Omega, \mathbf{S}^2)$, and v is the unique energy-minimizing map from Ω to $\mathbf{S}^2$ with $v|\partial\Omega \equiv \psi$. There exists a positive number β and, for any positive ε, a*

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