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LINEAR SYSTEMS ON K3-SECTIONS

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1. Introduction

The types of special linear systems which exist on a curve C which is a hyperplane section of a $K3$ surface X often do not depend on C but only on its linear equivalence class in X. For instance, Saint-Donat proved in [14] that C possesses a g^1_2 or g^1_3 if and only if the same is true for every nonsingular curve $C' \in |C|$, where $|C|$ denotes the linear system of C on X, and Reid [12] found some extensions of this result to other g^1_d's. The general question of whether the presence of a special g^r_d on a given hyperplane section C of a $K3$ surface forces the existence of such a g^r_d on every nonsingular $C' \in |C|$ arose out of work of Harris and Mumford [7]. Our purpose is to study this question and some related conjectures. We use the term $K3$-*section* to denote a smooth curve of genus at least two on a $K3$ surface. (Such a curve, if nonhyperelliptic, is a hyperplane section of a birational model of the $K3$ surface X in some projective embedding.)

We start, in §2, with a counterexample: a $K3$ surface X in $\mathbf{P}^{10}$, some of whose hyperplane sections (but not all) possess a g^1_4. In §3 we use a counting argument to show that if C carries a g^1_d which is scheme-theoretically isolated in moduli, then this g^1_d "propagates" to every nonsingular $C' \in |C|$, in the sense that an explicit geometric construction starting from the g^1_d on C produces a g^1_d on C'. A sufficient condition for the propagation of g^r_d's is also obtained, but it is weak for $r > 1$.

Analysis of our counterexample shows that in the family of all nonsingular hyperplane sections of X, the subfamily of curves carrying a g^1_4 has codimension one. On the other hand, all these curves *do* carry a g^2_6. Combining this observation with his theory of Koszul cohomology, Mark Green suggested that the correct conjecture is not propagation of g^r_d's but constancy of the "Clifford index" $\nu = d - 2r$. More precisely, for a line bundle M on a $K3$-section C with $h^0(M) = r + 1$, $\deg(M) = d$, and $\text{genus}(C) = g$, define

$$\nu(M) := d - 2r, \qquad \nu(C) := \min\{\nu(M) \mid r \geq 1,\ d \leq g - 1\}.$$

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