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PROPER FREDHOLM SUBMANIFOLDS OF HILBERT SPACE

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0. Introduction

The first step in the study of submanifolds of Euclidean spaces is to find enough local invariants and their relations so that they determine the submanifolds uniquely up to rigid motion. This is well known in classical differential geometry [7]. In fact, the first fundamental form I, the second fundamental form II and the induced normal connection are the basic local invariants, and they are related by the Gauss, Codazzi and Ricci equations. The shape operator A_v of an immersed submanifold M in $\mathbb{R}^n$ in the normal direction v at x is the selfadjoint operator on TM_x corresponding to the second fundamental form $\mathrm{II} \cdot v$. The eigenvalues of A_v are called the principal curvatures of M in the normal direction v. The Ricci equation implies that the normal curvature (the curvature of the normal connection) Ω^ν measures the commutativity of the shape operators, i.e., $\Omega^\nu(u, v) = [A_u, A_v]$. So if the normal curvature is zero, that is, if the normal bundle $\nu(M)$ is flat, then $\{A_v | v \in \nu(M)_x\}$ is a commuting family of selfadjoint operators, and locally there exists a parallel orthonormal normal frame field on M. It follows that many results of hypersurfaces can be generalized to submanifolds with flat normal bundles.

One natural type of problem is to determine all submanifolds of $\mathbb{R}^n$ which, in various senses, have simple local invariants. As a by-product of such investigations one often obtains many geometrically interesting examples of Riemannian manifolds. A special case of the above is the problem of finding all isoparametric submanifolds ([31], [30], [9], [41]), i.e., submanifolds with zero normal curvature and constant principal curvatures along any parallel normal field. It is not surprising that group theory provides examples. In fact, the principal orbits of the adjoint action of a simple Lie group on its Lie algebra (or more generally the principal orbits of the isotropy representations of symmetric spaces) are models for such manifolds. But they are still far from being completely classified.

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