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ON THE MEAN EXIT TIME FROM A MINIMAL SUBMANIFOLD

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Abstract

Let M^m be an immersed minimal submanifold of a Riemannian manifold N^n, and consider the Brownian motion on a regular ball $\Omega_R \subset M$ with exterior radius R. The mean exit time for the motion from a point $x \in \Omega_R$ is called $E_R(x)$. In this paper we find sharp support functions for $E_R(x)$ in the following distinct cases. The support is from below if the sectional curvatures of N are bounded from below by a nonnegative constant and the support is from above if the sectional curvatures of N are bounded from above by a nonpositive constant. It follows that the minimal submanifolds of $\mathbf{R}^n$ all have the same mean exit time function and we show that this function actually characterizes the minimal hypersurfaces in the set of all hypersurfaces of $\mathbf{R}^n$.

1. Introduction

Let M^m be an immersed (not necessarily minimal) submanifold of a complete Riemannian manifold N^n. The distance function on N is denoted by r so that $\text{dist}_N(p, x) = r_p(x)$ for all p, x in N. We now fix $p \in M \subset N$ and define a *regular domain* $\Omega_R(p) \subset M$ to be a smooth connected component of $B_R(p) \cap M^m$ which contains p. Here $B_R(p)$ is the geodesic R-ball around p in N subject to the usual restriction that $R < \min\{i_N(p), \pi\sqrt{\kappa}\}$, where κ is the supremum of the sectional curvatures of N, and $i_N(p)$ is the injectivity radius of N from p.

We now consider the Brownian motion on the domain Ω_R (cf. [4]) and denote by $E_R(x)$ the mean time of first exit from Ω_R for a Brownian particle starting at $x \in \Omega_R$. We want to compare $E_R(x)$ with the mean exit time function $\tilde{E}^b_R(\tilde{x})$ on the space form R-ball $\tilde{B}^m_R(\tilde{p})$ of dimension $m = \dim M$ and constant curvature $b \in \mathbf{R}$. Since $\tilde{B}_R$ has maximal isotropy at the center $\tilde{p}$, we have that $\tilde{E}_R$ only depends on the distance of $\tilde{x}$ from $\tilde{p}$. Hence we may, and do, write $\tilde{E}^b_R(\tilde{x}) = \tilde{\mathscr{E}}^b_R(r_{\tilde{p}}(\tilde{x})) = \tilde{\mathscr{E}}^b_R(r)$. In order to compare E and $\tilde{E}$ we transplant $\tilde{E}$ to Ω_R by the following definition:

$$\tilde{E}^b_R \colon \Omega_R \to \mathbf{R}; \quad \tilde{E}^b_R(x) = (\tilde{\mathscr{E}}^b_R \circ r_p)(x).$$

Our main result can now be formulated as follows.