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THE SYMPLECTIC STRUCTURE OF KÄHLER MANIFOLDS OF NONPOSITIVE CURVATURE

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Abstract

In this note we show that the Kähler form on a simply connected complete Kähler manifold W of nonpositive curvature is diffeomorphic to the standard symplectic form on $\mathbf{R}^n$. This means in particular that the symplectic structure on a Hermitian symmetric space of noncompact type is standard. We also show that if L is a totally geodesic proper, connected Lagrangian submanifold of a complete Kähler manifold W of nonpositive curvature, then W is symplectomorphic to the cotangent bundle T^*L with its usual symplectic structure provided that the fundamental group $\pi_1(W, L)$ vanishes. The proofs use a comparison theorem due to Greene & Wu and Siu & Yau.

1. Introduction

Let W be a Kähler manifold with a pole, i.e., a point p at which the exponential map is a diffeomorphism from the tangent space W_p onto W. Following [2], we will call a 2-dimensional subspace of the tangent space W_x a radial plane, if either $x = p$ or the subspace contains the tangent to the unique geodesic from p through x. The radial curvature of (W, p) is then the restriction of the sectional curvature function to the radial planes. Our first result is

Theorem 1. *Let (W, p) be a Kähler manifold with a pole such that the radial curvature is nonpositive. Then there is a diffeomorphism from W to $\mathbf{R}^n$ which takes the Kähler form ω on W to the standard symplectic form on $\mathbf{R}^n$.*

Note that any simply connected, complete Kähler manifold with nonpositive curvature satisfies the hypotheses. There are many such manifolds (see [1] for example). Observe also that the symplectomorphism which we construct from W to $\mathbf{R}^n$ is not in general holomorphic, for if it were it would preserve the Kähler metric.

Now suppose that L is a totally geodesic, connected and properly embedded Lagrangian submanifold of a complete Kähler manifold (W, ω) of nonpositive

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