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FLAT G-BUNDLES WITH CANONICAL METRICS

KEVIN CORLETTE

1. Introduction

A great deal of attention has recently been focused on the relationship between the invariant theory of semisimple algebraic group actions on complex algebraic varieties and the behavior of the moment map and quantum data provided by an associated symplectic structure. In particular, it has been shown that the moment map has zero as a regular value precisely when there are stable points, in the sense of geometric invariant theory [13]. This paper discusses an infinite dimensional instance of this correspondence involving flat bundles over compact Riemannian manifolds.

In finite dimensions, this philosophy received its simplest, and earliest, exposition in [11]. There, Kempf and Ness considered the case of a representation of a semisimple algebraic group G over $\mathbf{C}$ on a complex vector space V with some positive definite Hermitian form. Geometric invariant theory picks out a class of G-orbits called the stable ones. To be specific, $v \in V$ is stable if its orbit is closed in V and has the maximum possible dimension. Kempf and Ness observed that Gv is closed in V if and only if it contains a shortest vector. On the other side of the ledger, there is a symplectic structure on V associated with the chosen Hermitian form, and one has the action of the compact subgroup of G which preserves this form. There is a moment map associated with this action, and it turns out that the vanishing of this map at v is equivalent to v being the shortest vector in its orbit under G.

This relationship between symplectic geometry and algebraic geometry has been rephrased in the more sophisticated framework of geometric quantization in, for example, [9]. In another direction, Atiyah and Bott encountered an infinite dimensional instance of this correspondence in their study [1] of the Yang-Mills equations over Riemann surfaces. In that situation, the Kähler manifold in question was the space of all Hermitian connections on a Hermitian vector bundle over a compact Riemann surface. The analogue of G was the gauge group of vector bundle automorphisms. Atiyah and Bott

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