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POON'S SELF-DUAL METRICS AND KÄHLER GEOMETRY

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Abstract

It is shown that the self-dual conformal metrics on connected sums of $\mathbf{CP}_2$'s recently produced by Y. S. Poon arise from zero scalar curvature Kähler metrics on blow-ups of $\mathbf{C}^2$ by adding a point at infinity and reversing the orientation.

As noted by many authors ([4], [5], [6]), a complex surface with Kähler metric has anti-self-dual Weyl curvature iff the scalar curvature vanishes. On what would initially appear to be a completely unrelated front, Poon ([8], [9]) has produced positive scalar curvature self-dual metrics on connected sums of two and three complex projective planes. In fact, however, these phenomena are closely related:

Theorem. *Let $M = m\mathbf{CP}_2$, $0 \leq m \leq 3$, be equipped with a self-dual metric g of positive scalar curvature. There exists at least one point $p \in M$ such that $(M - \{p\}, g)$ is conformally isometric to $\mathbf{C}^2$ with m points blown up equipped with an asymptotically flat Kähler metric of zero scalar curvature.*

(**Remark.** The conformal isometry, of course, reverses orientation.)

Proof. Let $\pi: Z \to M$ be the canonical projection from the twistor space Z onto M; recall [1] that Z consists of all orthogonal almost-complex structure tensors on M inducing the reverse orientation. There exists ([8], [9]) a complex surface $\Sigma \subset Z$ isomorphic to $\mathbf{CP}_2$ blown up at m points such that $\pi|_\Sigma: \Sigma \to M$ is a diffeomorphism away from a projective line $L \subset \Sigma$ sent to a point $p \in M$; e.g. when $m = 0$, $M = S^4$, $Z = \mathbf{CP}_3$, and Σ is a hyperplane. By construction, $(\pi|_\Sigma)^*g$ is a Hermitian metric on $\Sigma - L$ but degenerates at L. Identifying $\Sigma - L$ with $\mathbf{C}^2$ blow up at m points, let

$$\hat{g} = (1 + r^2)^2(\pi|_\Sigma)^*g,$$

where r is the Euclidean distance from the origin in $\mathbf{C}^2$; this is not only Hermitian, but asymptotically flat, differing from the standard metric only by terms of order $1/r^2$ because the projection $\Sigma \to M$ is standard on the

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