J. DIFFERENTIAL GEOMETRY
28 (1988) 223-276

THE GEOMETRY OF LOOP GROUPS

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Abstract

The space ΩG of based loops on a compact Lie group admits a Kähler metric. Its curvature is expressed in terms of Toeplitz operators, and we define Chern classes by analogy with Chern-Weil theory in finite dimensions. In infinite dimensions extra geometric structure—a Fredholm structure—must be imposed before characteristic classes are defined. There is a natural Fredholm structure on ΩG induced from the family of Toeplitz operators. We use the index theorem for families of Fredholms parametrized by a group (proved in [20]) to show that the Chern classes of the Toeplitz family agree with the Chern classes defined by curvature. Explicit formulas for $\Omega \mathrm{SU}(n)$ are obtained. We also prove that the real characteristic classes of ΩG vanish for any group G. Extensions to more general groups of gauge transformations are considered.

Infinite dimensional geometry has received much attention recently, particularly due to motivations from physics. Rigorous consideration of infinite manifolds originated in the 1960's, when the foundations were carefully laid. Examples arising directly from variational problems in geometry and Lagrangian field theories in physics are manifolds of maps, and these can be modeled on Hilbert spaces. Exterior differential calculus, de Rham Theory, Riemannian connections, and all basic features of finite dimensional manifold theory generalize to Hilbert manifolds, but with one notable simplification: Hilbert manifolds are always parallelizable. This contrasts sharply with finite dimensionsm where twisted tangent bundles are ultimately due the nontrivial topology of $\mathrm{GL}(n;\mathbf{R})$. The structure group of a Hilbert manifold—the general linear group $\mathrm{GL}(\mathscr{H})$ on Hilbert space—is contractible, and the parallelizability of Hilbert manifolds is an immediate consequence.

Fredholm structures are reductions of the frame bundle to the topologically nontrivial group $\mathrm{GL}^{\mathrm{cpt}}(\mathscr{H})$ of invertible operators which differ from the identity by a compact operator [18]. This extra geometric structure was created to introduce twisting in the tangent bundle of a Hilbert manifold, but there seem to be few examples. Existence is not the issue, as a Hilbert manifold

Received January 7, 1987.