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ON THE SHAPE OF CANTOR SETS

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There has been much interest recently in sets which have a self-similar, fractal nature. As a prototype of a more general problem, we study a class of Cantor sets on the line from the point of view of bi-Lipschitz geometry, i.e., quasi-isometry. These investigations reveal a surprising general principle: A quasi-isometry between such objects is essentially the same thing as a map which is linear on the level of measure theory, i.e., has constant Radon-Nikodým derivative with respect to Hausdorff measure. This principle provides new invariants which enable us to classify generic Cantor sets of the type we consider.

Motivated by a question of Dennis Sullivan, we begin the classification of certain types of Cantor sets on the line. These Cantor sets each arise as the maximal invariant set of a map to the line defined on a neighborhood of the Cantor set. The middle third Cantor set arises in this way: Let $J_1 = [0, 1/3]$, $J_2 = [2/3, 1]$ and $J = [0, 1]$, and define $\tau\colon J_1 \cup J_2 \to J$ by

$$\tau(x) = \begin{cases} 3x & \text{if } x \in J_1, \\ 3x - 2 & \text{if } x \in J_2; \end{cases}$$

then $C = \tau^{-\infty}(J)$ is the middle third Cantor set. The ultimate goal is to understand the structure of C when τ is a locally expanding map. In this paper we consider the case when τ is defined on a finite number of disjoint closed intervals $J_1, \cdots, J_q$ contained in an interval J and τ maps each J_i affinely onto J. The topological and C^1 classifications of such sets are trivial, however the classification up to quasi-isometry (= order preserving bi-Lipschitz homeomorphism) gives a rich theory. The basic result is that a quasi-isometry can always be replaced by a quasi-isometry, possibly no longer surjective, with constant Radon-Nikodým derivative. The measure involved is the Hausdorff measure on the Cantor set, which is positive in its Hausdorff dimension. This result alone gives a powerful new invariant; previously the only known invariant was the Hausdorff dimension. These Cantor sets are homogeneous in the sense that any two points x, y have arbitrarily small affinely isomorphic neighborhoods in the Cantor set, although the isomorphism need not map x to y.

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