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A DIFFERENTIAL COMPLEX FOR POISSON MANIFOLDS

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0. Introduction

In this article, we consider Poisson manifolds M, that is, manifolds (say C^∞) for which there is a bracket operation $\{\ ,\ \}$ on smooth functions which has the usual properties of Poisson brackets. Poisson manifolds, apparently first considered by Lie, have been recently studied by Lichnérowicz [18] and by Weinstein [23]. Our main object here is the *canonical complex*

$$\cdots \to \Omega^{n+1}(M) \xrightarrow{\delta} \Omega^n(M) \xrightarrow{\delta} \Omega^{n-1}(M) \to \cdots,$$

where δ is given by the formula

$$\begin{aligned}\delta(f_0 f_1 \wedge \cdots \wedge df_n) &= \sum_{i=1}^{n}(-1)^{i+1}\{f_0, f_i\}\, df_1 \wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge df_n \\ &\quad + \sum_{1\le i<j\le n} (-1)^{i+j} f_0 d\{f_i, f_j\} \wedge df_1 \wedge \cdots \wedge \widehat{df_i} \\ &\qquad \wedge \cdots \wedge \widehat{df_j} \wedge \cdots \wedge df_n.\end{aligned}$$

This differential coincides with the one introduced by Koszul [17] which he denotes Δ.

The homology of the canonical complex is called the *canonical homology* of M. From its definition, it is clear there is a map from the *Lie algebra homology* $H_*(L, L)$, where L is the Lie algebra of C^∞-functions on M, with bracket $\{\ ,\ \}$, to the canonical homology of M.

The relation $d\delta + \delta d = 0$, proven by Koszul (where d denotes exterior differentiation), allows us to introduce a double complex, studied in §1.3.

In the case of symplectic manifolds, we prove in §§2.2 that δ is equal, up to sign, to $*d*$, where $*$ is the symplectic analog of the $*$ operator for Riemannian manifolds. We then conjecture that any de Rham cohomology class has a representative α such that $d\alpha = \delta\alpha = 0$. Some evidence for this conjecture is presented in §§2.2 and 2.3. We prove the conjecture for a compact Kähler

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