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THE SPLITTING THEOREM FOR SPACE-TIMES WITH STRONG ENERGY CONDITION

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1. Introduction

Our aim is the proof of the following theorem:

Theorem. *Let (M, g) be a connected, time oriented, globally hyperbolic Lorentzian manifold which is timelike geodesically complete ("tgc") and satisfies* $\mathrm{Ric}(v, v) \geqslant 0$ *for every timelike tangent vector v, where* Ric *denotes the Ricci tensor of g. Let $\gamma\colon \mathbb{R} \to M$ be a line, i.e. a timelike geodesic which realizes the distance between any two of its points. Then (M, g) is isometric to* $(\mathbb{R} \times H, -dt^2 \otimes h)$ *where (H, h) is a complete Riemannian manifold, and the factor* $(\mathbb{R}, -dt^2)$ *is represented by γ.*

This is the Lorentzian version of the Cheeger-Gromoll Splitting Theorem for Riemannian manifolds of nonnegative Ricci curvature [5] which solves a problem raised by S. T. Yau [12]. Our result extends earlier work of Galloway [8] and Beem et al. [2], [3]. Galloway [8] has proved the theorem under the additional assumption that M admits a smooth function whose level sets are compact spacelike Cauchy hypersurfaces. Beem, Ehrlich, Markvorsen, and Galloway [2], [3] proved a Toponogov type splitting theorem [10] for Lorentzian manifolds, i.e. they assumed $g(R(w, v)v, w) \geqslant 0$ for any timelike vector v and any $w \perp v$. The weaker Ricci curvature assumption of our theorem, called the strong energy condition (cf. [9]), is of particular interest in General Relativity. However, we need to assume the tgc property which can be concluded from the curvature assumption in the case of Beem et al.

The proofs of Cheeger and Gromoll [5] and H. Wu [11] for the Riemannian case apply the theory of elliptic operators to the Laplace operator Δ on the manifold. This fails in the Lorentzian case. In [6], a new proof was given which is based on the following idea: The Busemann functions b^+ and b^- of the given line γ satisfy $b^+ + b^- \leqslant 0$ with equality along γ, by the triangle inequality. On the other hand, by $\mathrm{Ric} \geqslant 0$, we have $\Delta b^\pm \geqslant 0$ in the sense of support

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