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# AN INDEX THEOREM ON OPEN MANIFOLDS. I

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## Introduction

This is the first of two papers which will describe and apply a new index theorem for elliptic operators on certain noncompact manifolds. This main result (Theorem 8.2) computes a real-valued index for operators of Dirac type on noncompact manifolds of "bounded geometry." In the sequel (Part II), several applications of this result will be given.

Let $D$ be a linear elliptic differential operator on a compact manifold $M$. Its kernel is then a finite-dimensional vector space of smooth functions, and one may try to compute the dimension of this space. This dimension depends rather sensitively on $D$, but the quantity

$$\operatorname{Ind}(D) = \dim(\ker D) - \dim(\operatorname{coker} D)$$

is a homotopy invariant of $D$, and the Atiyah-Singer index theorem [8], [4] calculates it from topological data.

Many analysts have produced generalizations of the index theorem to noncompact manifolds of various sorts, and these have been related to such diverse fields as the study of scalar curvature [23], number theory [5], representation theory [7], and the geometry of foliations [15]. Borrowing some terms from the theory of von Neumann algebras, we can roughly classify these results into three "types." Those of type I are nearest to the classical case. One imposes conditions sufficient to force the operator under consideration to be Fredholm in the usual sense, and the index is typically given by a formula similar to the usual one but with added correction terms. Under this heading could be included the $\eta$-invariant of Atiyah, Patodi, and Singer [6], the relative

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