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ON THE TOPOLOGY OF CLIFFORD ISOPARAMETRIC HYPERSURFACES

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A hypersurface in the unit sphere S^{n+1} is called *isoparametric* [5] if it has constant principal curvatures. The simplest, i.e., those for which the number g of distinct principal curvatures is less than or equal to 2, are the parallels of the equators and the product of spheres. Isoparametric hypersurfaces with $g = 3$ were classified by E. Cartan; they exist in dimensions $n = 3, 6, 12$, and 24. The above examples, being homogeneous, are well understood topologically. All the other isoparametric hypersurfaces have 4 or 6 distinct principal curvatures. Those with $g = 6$ exist only if $n = 6$ or 12.

Isoparametric hypersurfaces with $g = 4$ are the most interesting and have not been completely classified yet. With the exception of two cases in dimensions $n = 8$ and 18, all the known examples belong to the Clifford series discovered by Ferus, Karcher, and Münzner. For every orthogonal representation of the Clifford algebra C_{m-1} on $\mathbb{R}^l$, there corresponds [8] an isoparametric function on S^{2l-1} whose regular level sets are isoparametric hypersurfaces with $g = 4$. If $m \not\equiv 0 \pmod 4$, this function is determined by m and l up to a rigid motion of S^{2l-1}. If, however, $m \equiv 0 \pmod 4$, there are inequivalent representations of C_{m-1} on $\mathbb{R}^l$ parametrized by an integer q, the index of the representation. The unique (up to congruence) zero mean curvature (i.e., minimal) level set of an isoparametric function constructed from an index q representation of C_{m-1} on $\mathbb{R}^l$ is denoted by $M(m, l, q)$.

The aim of the present work is to study the topology of these hypersurfaces. We give a fairly complete classification of the $M(m, l, q)$ as well as their focal varieties in terms of homotopy, homeomorphism, and diffeomorphism types. For "small" l, the $M(m, l, q)$ are of distinct homotopy types, although their cohomological rings are the same. However, it turns out that the diffeomorphic types of $M(m, l, q)$ are periodic in q with a period d_m, the denominator of $B_{m/4}/m$, $B_{m/4}$ being the $(m/4)$th Bernoulli number.

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