J. DIFFERENTIAL GEOMETRY
27 (1988) 23–34

APPLICATION OF THE SELBERG TRACE FORMULA TO THE RIEMANN-ROCH THEOREM

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1. Introduction

The Riemann-Roch theorem is one of the foundational results in the theory of Riemann surfaces. Many proofs of this theorem are known, some traditional, others less so. The objective of this note is to give a proof in the latter vein, the primary tool being the Selberg trace formula.

Thus let X be a compact Riemann surface of genus $g > 1$. Then we may write $X = \Gamma \setminus \mathbf{D}$, $\mathbf{D}$ the open unit disk, Γ a discrete, strictly hyperbolic, cocompact subgroup of $H = G/\{\pm I\}$, $G = \mathrm{SU}(1,1)$. When applied to automorphic forms on Γ, the Riemann-Roch theorem and the Selberg trace formula say about the same thing. Consequently, it should not come as too much of a surprise that the one can be derived from the other.

For us, it will be convenient to regard the Riemann-Roch theorem as a statement about holomorphic line bundles on X. In turn, to get this into a group-theoretic context, it is necessary to use the language of automorphy factors. Once this transcription has been accomplished, it is technically simplest to pass to the $(g-1)$-fold covering group of G. Since the irreducible unitary representations of the universal covering group $\tilde{G}$ of G have been classified, no difficulty is encountered in doing so. Applying now the Selberg trace formula to suitable coefficients or quasi-coefficients then leads easily to the Riemann-Roch theorem.

It will be clear that what is said here can be said more generally. Nevertheless, we shall stay the course and not take up these side issues, interesting as they may be. Let us say only that Shimura [25] has proved a Riemann-Roch theorem for the traces of the Hecke operators. Agreeing to place ourselves in

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Received November 29, 1985 and, in revised form, December 1, 1986. The author's research was supported in part by the National Science Foundation.