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FLAT SPACETIMES

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Abstract

Those closed pseudo-Riemannian manifolds covered by Minkowski space M are classified up to finite covers. The simply transitive isometric actions on M are listed. Some spacetimes with 2 ends satisfying a causality condition are analyzed.

We call a 4-manifold X wtih a metric g of signature $(+,+,+,-)$ a spacetime. We will assume that g has zero curvature, i.e. X is flat. Then X is a special kind of affine manifold, namely an affine manifold with a parallel metric of the given signature. We also assume X is complete in the sense that geodesics on X extend for all time. This implies that the universal cover of X is Minkowski space M.

We will classify such X's under the assumption that X is compact. We prove in §1 that π has a solvable subgroup of finite index, using theory developed in [5] with W. Goldman. This result has been extended to higher dimensions by Goldman and Kamashima [6] but our proof is more geometric. For a noncompact counterexample see [8] and for further discussion see [9].

The classification also uses a theorem of Auslander's on unipotent simply transitive affine actions [1]. For subgroups of the isometry group $\mathscr{P}$ of Minkowski space, those are classified in §2. This is extended to all simply transitive actions in §3. Then in §4 we give our classification. It extends to dimension 4 that given by Auslander and Markus for 3-manifolds [2].

It is conceivable that if X is compact then g is automatically complete. A counterexample would be a very interesting spacetime: its curvature and global topology would not account for its failure to be complete. It would also be a valuable example in the theory of affine manifolds.

In §5 we discuss some two ended flat spacetimes with respect to their causal structure.

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