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THE TOTAL CURVATURE OF A KNOTTED TORUS

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Dedicated to James Eells, Jr. on his anniversary

0. Introduction

The classical total curvature of a simple closed smooth (C^∞)-curve γ in euclidean three-space $\mathbb{R}^3$ is

$$K(\gamma) = \int |\rho| \, ds,$$

where s is arclength and ρ is the curvature density. Denoting by $[\gamma]$ the *isotopy class of the knot*, define the classical total curvature of the isotopy type of γ as the greatest lower bound

$$K[\gamma] = \operatorname*{g.l.b.}_{\gamma' \in [\gamma]} K(\gamma').$$

Call the z-coordinate of a system of euclidean coordinates (x, y, z) the *height*. Then consider the number of critical points at relative maximal heights on any $\gamma' \in [\gamma]$ for which $z \mid \gamma'$ is nondegenerate. The minimum of these numbers for γ' isotopic to γ is called the *bridge index* $B[\gamma]$ of γ. Fox [4] gave lower bounds:

$$B[\gamma] \geqslant 1 + \sigma_1(\gamma) \geqslant 1 + \lambda(\gamma), \tag{0.1}$$

where $1 + \sigma_1(\gamma)$ is the minimal number of generators of the fundamental group of the complement $\mathbb{R}^3 \setminus \gamma$, and $\lambda(\gamma)$ is a number defined in Fox's differential calculus for knots and can be computed for any knot [1]. The invariants $\lambda(\gamma)$ and $B[\gamma] - 1$ (see [13]) are additive for connected sums of knots.

There exist knots for which $B(\gamma) > 1 + \sigma_1(\gamma)$, namely the torus knots $\gamma_{p,q}$, $p > q \geqslant 3$, with $B(\gamma) = q > 1 + \sigma_1 = 2$ [14], and knots for which $\sigma_1 \geqslant 1 > 0 = \lambda$ (see [13]).

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