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THE HEAT EQUATION SHRINKS EMBEDDED PLANE CURVES TO ROUND POINTS

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One can shorten a smooth curve immersed in a Riemannian surface by moving it in the direction of its curvature vector field. This process is known by many names, including "Curve Shortening," "Flow by Curvature," and "Heat Flow on Isometric Immersions." While this flow is defined by local information, it has many subtle and mysterious global properties. Even when the curve is immersed in the Euclidean plane, the global behavior is very difficult to analyze. Most striking are the facts that a convex curve shrinks to a point, becoming round in the limit, and that, in the absence of singularities, embedded curves remain embedded. We will add to this list the fact that embedded curves become convex without developing singularities. This fact completes the proof of the conjecture that curve shortening shrinks embedded plane curves smoothly to points, with round limiting shape.

The Main Theorem. *Let* $C(\cdot, 0)\colon S^1 \to \mathbf{R}^2$ *be a smooth embedded curve in the plane. Then* $C\colon S^1 \times [0, T) \to \mathbf{R}^2$ *exists satisfying*

$$\partial C/\partial t = \kappa \cdot \mathbf{N}, \tag{$*$}$$

where K *is the curvature of* C, *and* $\mathbf{N}$ *is its unit inward normal vector.* $C(\cdot, t)$ *is smooth for all* t, *it converges to a point as* $t \to T$, *and its limiting shape as* $t \to T$ *is a round circle, with convergence in the* C^∞ *norm.*

A more visual description of this flow is the evolution of elastic bands in honey. If the tension in the elastic is kept constant, then its behavior is determined (approximately) by equation (∗). For a discussion of this problem in its most general setting, the reader is referred to [5].

For the case where the initial curve is convex, this theorem was proven by M. Gage and R. Hamilton in [5].

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