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A GENERALIZATION OF BERGER'S THEOREM ON ALMOST 1/4-PINCHED MANIFOLDS. II

O. DURUMERIC

1. Introduction

Let (M^n, g) be a compact, smooth Riemannian manifold and let $K(M, g)$, $d(M, g)$, $i(M, g)$, and $(\tilde{M}, \tilde{g})$ denote its sectional curvature, diameter, injectivity radius, and Riemannian universal cover, respectively. In this paper, we investigate Riemannian manifolds of positive sectional curvature. For normalization, we take $K(M, g) \geqslant 1$. Let $S^n(1)$, $\mathbf{R}P^n(1)$, $\mathbf{C}P^n$, $\mathbf{H}P^n$, $\mathrm{Ca}P^2$ denote the standard sphere of radius one, the projective spaces on real, complex numbers, and quaternions, and the Cayley plane with their standard metrics, respectively. $S^n(1)$ and $\mathbf{R}P^n(1)$ have constant sectional curvature 1, while the rest have $1 \leqslant K(\cdot) \leqslant 4$. The diameter of $S^n(1)$ is π, and the rest have diameter $\pi/2$. These Riemannian manifolds, except $\mathbf{R}P^n(1)$, are all of the compact simply connected symmetric spaces of rank 1, up to a constant factor of the metric.

If $K(M, g) \equiv 1$, then $(\tilde{M}, \tilde{g})$ is isometric to $S^n(1)$ [37, p. 69]. By the classical Sphere Theorem [1], [26], [7]: If $1 \leqslant K(M, g) < 4$, then $\tilde{M}$ is homeomorphic to S^n. This result is optimal by the examples above. In [1], M. Berger proved the rigidity theorem: If $1 \leqslant K(M, g) \leqslant 4$, then either $\tilde{M}$ is homeomorphic to S^n or $(\tilde{M}, \tilde{g})$ is isometric to a symmetric space of rank 1. Recently, M. Berger obtained that for even n, there exists a universal constant $\varepsilon(n) > 0$ depending only on n such that if $1 \leqslant K(M^n, g) \leqslant 4 + \varepsilon(n)$, then either $\tilde{M}^n$ is homeomorphic to S^n or diffeomorphic to $\mathbf{C}P^{n/2}$, $\mathbf{H}P^{n/4}$, or $\mathrm{Ca}P^2$ [2].

Some generalizations of the above were given involving the diameter of (M, g). Bonnet: If $K(M, g) \geqslant 1$, then $d(M, g) \leqslant \pi$ [7, p. 27]. The rigidity for the maximal diameter is obtained by Toponogov: If $K(M, g) \geqslant 1$ and

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