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CONSTRUCTION OF CONNECTION INDUCING MAPS BETWEEN PRINCIPAL BUNDLES. PART I

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0. Introduction

Consider two C^∞-smooth principal bundles, say $X \to V$ and $Y \to W$, with the same structure group G and with C^∞ connections Γ on X and Δ on Y, respectively. We look for a C^∞-map $f: V \to W$ such that the induced bundle $f^*(Y)$ over V with the induced connection $f^*(\Delta)$ is isomorphic to (X, Γ). This means that f can be covered by (or lifted to) a morphism of bundles, $F: X \to Y$, inducing Γ from Δ, which is expressed by $F^*(\Delta) = \Gamma$.

0.1. The problem of inducing connections was first studied by Narasimhan and Ramanan [3] who proved that for a given compact Lie group G and an integer $n = 0, 1, \cdots$, there exists a (universal) bundle (Y, Δ) over some (classifying) compact manifold W, such that every G-bundle X over an n-dimensional manifold V with an arbitrary C^∞-connection Γ can be induced by a C^∞-morphism $F: X \to Y$. Furthermore, they give a precise description of the universal connection Δ for the unitary and the orthogonal groups. Namely, if $G = \mathrm{U}(p)$ they take the Grassmann manifold $\mathrm{Gr}_p(\mathbf{C}^q)$ for W and use the standard connection Δ on the canonical bundle $Y \to \mathrm{Gr}_p(\mathbf{C}^q)$ (here Y is the Stiefel manifold of orthogonal p-frames in $\mathbf{C}^q$). The dimension q for which they prove the existence of F is $q = (n+1)(2n+1)p^3$, where $n = \dim V$. Similarly, for $G = \mathrm{O}(p)$, their method provides a connection inducing map into the real Grassmann manifold $\mathrm{Gr}_p(\mathbf{R}^q)$, again for $q = (n+1)(2n+1)p^3$.

0.2. The result by Narasimhan-Ramanan was improved for $G = \mathrm{O}(p)$ by Gromov (see 2.2.6 in [1]) who showed the existence of a connection inducing map $f: V \to \mathrm{Gr}_p(\mathbf{R}^q)$ for $q = \max(p(n+1), p(n+2)+n)$. Furthermore, if the manifold V is parallelizable and the bundle $X \to V$ is trivial, then

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