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# RELATED ASPECTS OF POSITIVITY IN RIEMANNIAN GEOMETRY

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## 1. Introduction

There are several numerical functions which can be related in the geometric context of Riemannian manifold, especially those which are complete and have constant negative curvature. They are:

(i) The Hausdorff dimension $D(X)$ of a closed set $X$ in Euclidean space.

(ii) The point $\lambda_0$ of the $L^2$-spectrum nearest to zero for the Friedrich extension of a semidefinite symmetric operator $\Delta$.

(iii) The critical exponent $\delta(\Gamma)$ of the Poincaré-Dirichlet series of a discrete group $\Gamma$ of Moebius transformations of $S^d$.

(iv) The parameter of the spherical complementary series of irreducible representations of $0(n,1)$, especially $PSl(2,\mathbf{R})$ and $PSl(2,\mathbf{C})$.

(v) The exponential rate of transience of a positivity preserving Markoff operator $P$, equivalently the point nearest to zero in the 'positive spectrum' of a Markoff operator $P$, i.e., $\lambda$-potential theory.

(vi) The entropy of an ergodic measure preserving flow.

A rich supply of examples is given by groups $\Gamma$ of non-Euclidean motions of $\mathbf{H}^{d+1}$ having *finite sided* fundamental domains in $\mathbf{H}^{d+1}$ whose Poincaré limit set in $S^d$ has Hausdorff dimension $D > \frac{1}{2}d$. Then:

(1) The Hausdorff measure $\mu$ of the limit set is finite and positive (when the ranks of any cusps are at most $D$).

(2) $\mu$ on the $\lambda$-Martin boundary $S^d$ of $\mathbf{H}^{d+1}$ defines a positive $\lambda$-eigenfunction $\phi_\mu$ for the Laplacian which is $\Gamma$-invariant, where $\lambda = D(d-D)$. (In the situation of (1), $\phi_\mu(p)$ has the following nice geometrical interpretation: It is the total Hausdorff mass of the limit set as viewed from $p$.)

(3) $\phi_\mu$ belongs to $L^2(\mathbf{H}^{d+1}/\Gamma)$ and is the lowest eigenfunction or ground state.

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