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GEODESIC LENGTH FUNCTIONS AND THE NIELSEN PROBLEM

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0. Introduction

Fundamental to the synthetic geometry of a space is the behavior of geodesics. The geometry of a noncomplete metric can be rather disappointing: points may fail to be joined by geodesics and length minimizing curves may fail to exist. The Weil-Petersson metric for Teichmüller space is not complete [21]. Nevertheless we shall show that its synthetic geometry is quite similar to that of a complete metric of negative curvature. Our main result is that every pair of points is joined by a unique geodesic. Also we find that Teichmüller space has an exhaustion by compact Weil-Petersson convex sets. In particular the exponential map is a homeomorphism from its domain to Teichmüller space, the analogue of the Hadamard-Cartan theorem, and furthermore the exponential map is distance increasing, a standard result for complete negative curvature metrics. We also show that a finite group of isometries has a fixed point, the analogue of the Cartan center of mass result. A solution of the Nielsen problem is an immediate corollary (see §6): every finite subgroup of the mapping class group fixes a point of Teichmüller space [12].

For the study of geodesics completeness is used to bound sequences and arcs away from infinity in the one-point compactification of a space. Our approach is to substitute a natural class of proper convex functions for completeness. The functions are the geodesic length functions introduced by Fricke-Klein [7], later studied by Fenchel-Nielsen [6], Keen [11], Kerckhoff [12], [13], Thurston [19], the author [22], [23], [24], [27], as well as others. Our main result on this topic is that a geodesic length function is strictly convex along a Weil-Petersson geodesic. In fact our results for the Weil-Petersson metric are based on three observations: that proper geodesic length functions exist, their

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