J. DIFFERENTIAL GEOMETRY
25 (1987) 249–273

ON SURFACES WITH NO CONJUGATE POINTS

W. BALLMANN, M. BRIN & K. BURNS

A complete Riemannian manifold M has no conjugate points if any two points in its universal cover are joined by a unique geodesic. If the sectional curvature of M is nonpositive, then M has no conjugate points; the converse is not true even for compact surfaces. A natural question is to what extent properties of manifolds of nonpositive sectional curvature are valid for manifolds with no conjugate points. For example, by the Gauss-Bonnet theorem, any metric of nonpositive curvature on the torus T^2 is flat. In 1943, E. Hopf [4] proved

Theorem. *Any metric on T^2 with no conjugate points is flat.*

The best way to explain the purpose of our paper and to introduce the necessary notations is to give an outline of Hopf's argument. He considers the Riccati equation

$$u' + u^2 + K(\gamma_v(t)) = 0, \tag{0.1}$$

where γ_v is the geodesic with initial velocity v and K is the Gaussian curvature. Denote by $u_R^+(v, \cdot)$ (resp. $u_R^-(v, \cdot)$) the solution of (0.1) which satisfies $u_R^+(v, -R) = +\infty$ (resp. $u_R^-(v, R) = -\infty$). Note that $u_R^+(v, t)$ is the geodesic curvature at $\gamma_v(t)$ of the geodesic circle of radius $t + R$ centered at $\gamma_v(-R)$. As we explain in the next section (see Proposition 1.3), if a surface S has no conjugate points, then $u_R^+(v, t)$ (resp. $u_R^-(v, t)$) is defined for $t > -R$ (resp. $t < +R$) and there are well-defined limit solutions

$$u^+(v, \cdot) = \lim_{R\to\infty} u_R^+(v, \cdot), \qquad u^-(v, \cdot) = \lim_{R\to\infty} u_R^-(v, \cdot).$$

Recall that the geodesic flow g^t acts on the unit tangent bundle T_1S by $g^tv = \dot\gamma_v(t)$ and preserves the Liouville measure $d\mu = dA \times d\lambda$, where A is area in S and λ is Lebesgue measure on the unit circle. The solutions u^+ and u^- are invariant under g^t in the following sense:

$$u^\pm(g^tv, s) = u^\pm(v, s + t). \tag{0.2}$$

Received April 21, 1986. The first author was partially supported by National Science Foundation Grant #DMS-85-03742; the second and third authors were partially supported by National Science Foundation Grant #DMS-82-04024.