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# INVARIANTS OF CONFORMAL LAPLACIANS

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The conformal Laplacian $\square = d^*d + (n-2)s/4(n-1)$, acting on functions on a Riemannian manifold $M^n$ with scalar curvature $s$, is a conformally invariant operator. In this paper we will use $\square$ to construct new conformal invariants: one of these is a pointwise invariant, one is the integral of a local expression, and one is a nonlocal spectral invariant derived from functional determinants.

We begin in §1 by describing the Laplacian $\square$ and its Green function in the context of conformal geometry. We then derive a basic formula giving the variation in the heat kernel of $\square$. This formula is strikingly simpler than the corresponding formula for the ordinary Laplacian given by Ray and Singer [15].

The heat kernel of $\square$ has an asymptotic expansion $k(t,x,x) \sim (4\pi t)^{-n/2}\sum a_k(x)t^k$. In §2 we prove that $a_{(n-2)/2}$ is a pointwise conformal invariant of weight $-2$, i.e. it satisfies $a_{(n-2)/2}(x;\lambda^2 g) = \lambda^2 a_{(n-2)/2}(x;g)$, where $g$ is the metric and $\lambda$ is any smooth positive function. In particular, this shows the existence of a nontrivial locally computable conformally invariant density naturally associated to the conformal structure of an even dimensional manifold. The key to the proof is to consider the parametrix of the Green's function, which is obtained from the heat kernel by an integral transform. One finds that $a_{(n-2)/2}$ occurs as the coefficient of the first log term in this parametrix, and its conformal invariance then follows directly from the conformal invariance of the Green's function.

In §3 we show that $\int a_{n/2}$ is a global conformal invariant (the calculations in §4 show that it is not a pointwise invariant). The proof is a direct calculation of the invariant of $\int a_{n/2}$ using equation (1.10).

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