J. DIFFERENTIAL GEOMETRY
25 (1987) 139–156

COLLAPSING RIEMANNIAN MANIFOLDS TO ONES OF LOWER DIMENSIONS

KENJI FUKAYA

0. Introduction

In [7], Gromov introduced a notion, Hausdorff distance, between two metric spaces. Several authors found that interesting phenomena occur when a sequence of Riemannian manifolds M_i collapses to a lower dimensional space X. (Examples of such phenomena will be given later.) But, in general, it seems very difficult to describe the relation between topological structures of M_i and X. In this paper, we shall study the case when the limit space X is a Riemannian manifold and the sectional curvatures of M_i are bounded, and shall prove that, in that case, M_i is a fiber bundle over X and the fiber is an infranilmanifold. Here a manifold F is said to be an infranilmanifold if a finite covering of F is diffeomorphic to a quotient of a nilpotent Lie group by its lattice.

A complete Riemannian manifold M is contained in class $\mathscr{M}(n)$ if **dim** $M \leqslant n$ and if the sectional curvature of M is smaller than 1 and greater than -1. An element N of $\mathscr{M}(n)$ is contained in $\mathscr{M}(n, \mu)$ if the injectivity radius of N is everywhere greater than μ.

Main Theorem. *There exists a positive number $\varepsilon(n, \mu)$ depending only on n and μ such that the following holds.*

If $M \in \mathscr{M}(n)$, $N \in \mathscr{M}(n, \mu)$, and if the Hausdorff distance ε between them is smaller than $\varepsilon(n, \mu)$, then there exists a map $f\colon M \to N$ satisfying the conditions below.

(0-1-1) *(M, N, f) is a fiber bundle.*

(0-1-2) *The fiber of f is diffeomorphic to an infranilmanifold.*

(0-1-3) *If $\xi \in T(M)$ is perpendicular to a fiber of f, then we have*

$$e^{-\tau(\varepsilon)} < |df(\xi)|/|\xi| < e^{\tau(\varepsilon)}.$$

Received October 21, 1985 and, in revised form, April 11, 1986.