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# STRANGE ACTIONS OF GROUPS ON SPHERES

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A theme in topology is that certain group actions may be made geometric by a change of coordinates. In this paper geometric means conformal. In the mid 1970's F. Gehring and B. Palka expressed hope that a uniformly quasiconformal action $G \times S^n \to S^n$ is conjugate by a quasiconformal homeomorphism to a conformal action [11]. This was proved to be true by D. Sullivan [20] and P. Tukia [21] when $n = 2$.

Let $F_r$ denote a free group of rank $r$ and $F_r \rtimes \mathbf{Z}_{2r}$ a certain semidirect product (defined precisely later). One of our two main results is (see §3): For $r$ sufficiently large[1] there is a discrete, smooth, uniformly quasiconformal action $\psi$: $(F_r \rtimes \mathbf{Z}_{2r}) \times S^2 \to S^3$ which is not conjugate (by any homeomorphism) to a conformal action.

There has been interesting earlier work in this direction. Tukia [22] for $n > 2$ constructed a uniformly quasiconformal action $G \times S^n \to S^n$ of a connected solvable Lie group $G$, where $G$ does not embed in the Möbius group of $S^n$. Our example differs from Tukia's in that our action is discrete and smooth ($= C^\infty$). Recently, G. Martin [15] has constructed a discrete (but not smooth), uniformly quasiconformal action on $S^n$, $n \geqslant 3$, which is not quasiconformally conjugate to a conformal action but is topologically conjugate to a conformal action.

The failure of the higher dimensional Smith conjecture is relevant. It was long known to topologists that for each $n \geqslant 4$ there are smooth, finite cyclic actions on $S^n$ whose fixed point sets are nontrivially knotted $(n-2)$-spheres [12]. These, of course, could not be topologically conjugate to elliptic (conformal) groups which after a further conjugation are linear. In fact, the action $\psi$

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[1] The minimal $r$ suitable in our constructions seems to be more than ten and less than 100.