J. DIFFERENTIAL GEOMETRY
25 (1987) 1–22

FUNDAMENTAL GROUPS OF MANIFOLDS OF NONPOSITIVE CURVATURE

WERNER BALLMANN & PATRICK EBERLEIN

Introduction

The universal covering space H of a complete Riemannian manifold M of nonpositive sectional curvature is diffeomorphic to $\mathbf{R}^n$, $n = \dim(M)$. Hence the homotopy type of M is completely determined by the isomorphism class of the fundamental group Γ of M. It is, therefore, only natural to expect strong relations between the geometric structure of M and the algebraic structure of Γ. In this paper we obtain several such relations:

A general assumption in the results we state below is that

(1) the sectional curvature is nonpositive and bounded from below by some constant $-a^2$ and

(2) the volume of M is finite.

We define the rank of a unit tangent vector v of M, rank(v), to be the dimension of the space of all parallel Jacobi fields along the geodesic γ_v which has initial velocity v. The minimum of rank(v) over all $v \in SM$ is called the rank of M. This agrees with the usual rank if M is a locally symmetric space. Manifolds of rank one resemble manifolds of strictly negative curvature (see [2] [3], and §2 below). Manifolds of higher rank are studied in [5], [6], [7], and [3], and the conclusive result is that H is a space of rank one, or a symmetric space, or a Riemannian product of such spaces. This is the basic ingredient in the proofs of our results; it allows us, more or less, to consider only the cases that H is of rank one or a symmetric space.

As a first example of this principle we indicate in our preliminary section the proof of the following theorem.

Theorem A. *Either M is flat or Γ contains a nonabelian free subgroup.*

This is an improvement of the result of Avez [1] that Γ has exponential growth if M is compact and not flat.

Received July 15, 1985 and, in revised form, March 4, 1986. The authors were partially supported by National Science Foundation Grants DMS-8503742 and MCS-8219609, respectively.