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CONNECTED COMPONENTS OF MODULI SPACES

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0. Introduction

Let S be a minimal surface of general type (complete and smooth over $\mathbb{C}$), and let $\mathcal{M} = \mathcal{M}(S)$ (resp., $\mathcal{M}^{\text{diff}}$) be the coarse moduli space of complex structures on the oriented topological (resp., differential) 4-manifold underlying S.

By Gieseker's theorem [5], $\mathcal{M}(S)$ is a quasiprojective variety, and the number $\nu(S)$ of its irreducible components is bounded by a function $\nu_0(K^2, \chi)$ of the two (topological) invariants $K^2 = K_S^2$, $\chi = \chi(\mathcal{O}_S)$.

Let $\lambda(S)$ be the number of connected components of $\mathcal{M}(S)$: this short note answers a question raised in a previous paper [1], showing that the above number $\lambda(S)$ can be arbitrarily large.

As in [1], to which we shall constantly refer, again we restrict our attention to bidouble (i.e., Galois with group $(\mathbb{Z}/2)^2$) covers of $Q = \mathbb{P}^1 \times \mathbb{P}^1$: indeed, (cf. [2]) we conjecture a stronger result to hold true, namely that many of the different irreducible components of $\mathcal{M}$ we thus obtain are in fact connected components of $\mathcal{M}$.

The idea of proof is rather simple: if S and S' are deformations of each other, then there exists a diffeomorphism $f: S \to S'$ such that $f^*(K_{S'}) = K_S \in H^2(S, \mathbb{Z})$, and, in particular, if $r(S) = \max\{r \in \mathbb{N} \mid (1/r)K_S \in H^2(S, \mathbb{Z})\}$, then $r(S) = r(S')$.

In view of Donaldson's recent result [3], it is possible that the integer $r(S)$ could be an invariant of the differentiable structure for these surfaces; it is not clear at the moment whether nicer properties are enjoyed by the moduli spaces $\mathcal{M}^{\text{diff}}(S)$. Nevertheless, when the complex dimension is at least 3, it seems (cf. [6], [7]) that similar phenomena of high disconnectedness should appear also for $\mathcal{M}^{\text{diff}}$.

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