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CONNECTIONS, COHOMOLOGY AND THE INTERSECTION FORMS OF 4-MANIFOLDS

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I. Introduction

The Yang-Mills gauge fields, which were first introduced in Mathematical Physics, can be used to obtain strong results about the differential topology of 4-manifolds. In a previous paper [12] simply connected manifolds with definite quadratic intersection form were studied through the associated moduli spaces of "instanton" solutions to the Yang-Mills differential equations. Here we shall extend these methods to discuss the existence of smooth, simply connected, 4-manifolds with certain indefinite intersection forms. A companion article [14] will discuss non-simply-connected manifolds (about which results have recently been obtained by Fintushel and Stern [16]) and we shall develop here a number of techniques to be used in that article and also in other applications [13].

If X is a closed oriented 4-manifold, then the intersection of 2-cycles defines a unimodular bilinear form on the free group: $H_2(X; \mathbb{Z})/\text{Torsion}$. Changing the orientation of the 4-manifold reverses the sign of the form and we shall adopt here the opposite convention to [12], that is, eventually we consider "anti-self-dual" connections—this fits in better with the conventional orientation of complex surfaces. With this convention the Theorem of [12] becomes:

(1.1) Theorem A. *If X^4 is smooth, compact, and simply connected and with negative intersection form ($\alpha \cdot \alpha \leqslant 0$ for all α in H_2), then the form is equivalent over the integers for the standard example*:

$$(-1) \oplus (-1) \oplus \cdots \oplus (-1).$$

Of course, the point of this is that many nonstandard definite forms exists; for example, the positive definite root matrix E_8 and its multiples $\pm n \cdot E_8$.

The nonsingular forms over the real numbers are classified by rank and the number b^+ of positive eigenvalues in a diagonalization. According to the Hasse-Minkowski classification [23] the only other invariant for indefinite

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