J. DIFFERENTIAL GEOMETRY
24 (1986) 153-179

FOUR-MANIFOLDS WITH POSITIVE CURVATURE OPERATOR

RICHARD S. HAMILTON

1. A compact surface with positive mean scalar curvature must be diffeomorphic to the sphere S^2 or the real projective space RP^2. A compact three-manifold with positive Ricci curvature must be diffeomorphic to the sphere S^3 or a quotient of it by a finite group of fixed point free isometries in the standard metric, such as the real projective space RP^3 or a lens space $L^3_{p,q}$. This was proven in [1]. Our main result is the following generalization to four dimensions.

1.1. Theorem. *A compact four-manifold with a positive curvature operator is diffeomorphic to the sphere S^4 or the real projective space RP^4.*

Here we regard the Riemannian curvature tensor $\mathrm{Rm} = \{R_{ijkl}\}$ as a symmetric bilinear form on the two-forms Λ^2 by letting

$$\mathrm{Rm}(\phi, \psi) = R_{ijkl}\phi_{ij}\psi_{kl}.$$

We say the manifold has a positive curvature operator if $\mathrm{Rm}(\phi, \phi) > 0$ for all two-forms $\phi \neq 0$, and a nonnegative curvature operator if $\mathrm{Rm}(\phi, \phi) \geqslant 0$ for all ϕ.

These results extend to the case of nonnegative curvature. A compact surface with nonnegative mean scalar curvature must be diffeomorphic to a quotient of the sphere S^2 or the plane R^2 by a group of fixed-point free isometries in the standard metrics. The examples are the sphere S^2, the real projective space RP^2, the torus $T^2 = S^1 \times S^1$, and the Klein bottle $K^2 = RP^2 \# RP^2$ (where # denotes the connected sum).

1.2. Theorem. *A compact three-manifold with nonnegative Ricci curvature is diffeomorphic to a quotient of one of the spaces S^3 or $S^2 \times R^1$ or R^3 by a group of fixed point free isometries in the standard metrics.*

The quotients of $S^2 \times R^1$ include $S^2 \times S^1$, $RP^2 \times S^1$, the unoriented S^2 bundle over S^1, and the connected sum $K^3 = RP^3 \# RP^3$. The quotients of R^3 are the torus T^3 and five other flat three-manifolds.

Received September 27, 1985.