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HARMONIC MAPS OF THE TWO-SPHERE INTO THE COMPLEX HYPERQUADRIC

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Introduction

Let $G(k, n;\ \mathbf{C})$ denote the Grassmann manifold of all k-dimensional subspaces $\mathbf{C}^k$ of complex n-space $\mathbf{C}^n$. Let P_{n-1} denote complex projective $(n-1)$ space, $P_{n-1} = G(1, n;\ \mathbf{C})$ and let $Q_{n-2} \subset P_{n-1}$ denote the complex hyperquadric, that is, the complex hypersurface, of P_{n-1} defined by the equation

$$Z_0^2 + Z_1^2 + \cdots + Z_{n-1}^2 = 0,$$

where $\{Z_0, \cdots, Z_{n-1}\}$ are homogeneous coordinates of P_{n-1}. Q_{n-2} has a natural Kähler metric which it inherits as a complex submanifold of P_{n-1}. In this note we will study the minimal immersions or harmonic maps of the two-sphere S^2 into Q_{n-2}. Our result can be described as follows: To each harmonic map $f\colon S^2 \to Q_{n-2}$ we associate a *directrix curve* $\Delta_f\colon S^2 \to G(2, n;\ \mathbf{C})$ which is either a holomorphic curve or a degenerate harmonic map. (The degenerate harmonic maps arise in the study of harmonic maps $S^2 \to G(2, n;\ \mathbf{C})$. In [4] it is shown that they can be constructed from holomorphic curves $S^2 \to P_{n-1}$.) The directrix curve Δ_f will be shown to satisfy strong nullity conditions, in the sense that its lth osculating space is null for $0 \leqslant l \leqslant r$ (where $r \geqslant 0$ depends on f). The harmonic map f can be recovered from its directrix curve Δ_f via differentiation and the choice of holomorphic sections of P_1 bundles over S^2. This description and Calabi's description of minimal maps $S^2 \to S^N$ [1] are related. In fact, the nullity conditions on the directrix curves of harmonic maps $S^2 \to Q_{n-2}$ are similar to those on the directrix curves of minimal maps $S^2 \to S^N$.

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