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A UNIVERSAL SMOOTHING OF FOUR-SPACE

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Except in dimension four, smooth structures can be classified up to ε-isotopy by bundle reductions. Since R^n is contractible, this implies that any smooth structure Γ on R^n, $n \neq 4$, is ε-isotopic to the standard one. In contrast, R^4 has many distinct smoothings (even up to diffeomorphism.) We construct a certain smoothing of the half-space, $\frac{1}{2}R^4 = \{(x_1, x_2, x_3, x_4) | x_4 \geqslant 0\}$ and write H for this half-space together with its smooth structure. H contains all other smoothings of $\frac{1}{2}R^4$ (see Theorem 1) and is unique with respect to this property (Corollary A). H is the universal half-space. The interior $\dot{H} = U$ is naturally identified (replace x_4 with $\ln x_4$) with a smoothing of R^4. Corollary B states that U contains every smoothing of R^4 imbedded within it. Thus, we say U is a universal R^4. The construction of U is unambiguous but we do not claim that any R^4_Γ into which all smooth R^4's imbed is diffeomorphic to U. This is not known.

Copies of H may be used to "corrupt" the differential structure of any open manifold near its ends. This process irons out any unstable differences (i.e., those not persisting into dimension five under product with the real line) in differentiable structure and may be thought of as giving the unique worst structure (in a given stable class) on the manifold (Theorems 2 and 3).

Theorem 4 says that U cannot arise as the interior of a topologically flat cell in any smooth 4-manifold. This nonimbedding result leads, by an observation of R. Gompf, to countably many structures on R^4 [7]. A somewhat formal derivation of this is given in the proof of Corollary D.

In smooth 4-manifold topology, Casson handles [1], [5] are often constructable from homotopy (or ε-homotopy) information where smoothly imbedded disks are not present. The proof of Quinn's 5-dimensional controlled h-cobordism theorem [10] may be followed just to the point where smoothness is about to be lost. This is where isotopies are defined along Casson handles

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