J. DIFFERENTIAL GEOMETRY
24 (1986) 15–17

THE LINEARITY OF PROPER HOLOMORPHIC MAPS BETWEEN BALLS IN THE LOW CODIMENSION CASE

JAMES J. FARAN

Let $B^n = \{z \in C^n\colon \|z\| < 1\}$ and let $f\colon B^n \to B^k$ be a proper holomorphic map. We shall always take $n > 2$. Cima and Suffridge [1] have conjectured that if f extends to a twice continuously differentiable function on the closure of B^n and $k \leqslant 2n-2$, then f is linear fractional. The purpose of this note is to show

Theorem. *If $f\colon B^n \to B^k$ is a proper holomorphic map which extends holomorphically to a neighborhood of $\overline{B^n}$ and $k \leqslant 2n-2$, then f is linear fractional.*

(It should be remarked that the map $(z_1,\cdots,z_n) \to (z_1,\cdots,z_{n-1}, z_1z_n,\cdots,z_{n-1}z_n, z_n^2)$ shows that the theorem is false if $k \geqslant 2n-1$; see [1].)

So, let $f\colon B^n \to B^k$ be a proper map, holomorphic in a neighborhood of $\overline{B^n}$, $k \leqslant 2n-2$. Let $\langle z,w\rangle = \sum_{j=1}^{p} z_j\overline{w}_j$ be the hermitian inner product in C^p. Let $z' = f(z)$. Applying the Hopf lemma to the function $r' = \langle z',z'\rangle - 1$ on B^n, we see that

$$\langle z',z'\rangle - 1 = u(z,\bar{z})(1 - \langle z,z\rangle) \tag{1}$$

for some real analytic function $u(z,\bar{z})$, nonzero in a neighborhood of ∂B^n. Complexifying, (1) becomes

$$\langle z',w'\rangle - 1 = u(z,\overline{w})(1 - \langle z,w\rangle), \tag{2}$$

where $w' = f(w)$.

Let $z_0 \in \partial B^n$. (2) is valid for $(z,w) \in U \times U$ for some open neighborhood U of z_0. Thus if z is a point on the hyperplane $Q_w = \{\zeta\colon 1 - \langle \zeta,w\rangle = 0\}$, $(z,w) \in U \times U$, then $z' = f(z)$ is on the hyperplane $Q'_{w_0} = \{\zeta'\colon 1 - \langle \zeta',w'\rangle = 0\}$, $w' = f(w)$. Thus f maps points lying in a complex hyperplane to points lying in a complex hyperplane. Let $\phi_n\colon P^n \to P^{n*}$ be the antiholomorphic map

Received April 16, 1984 and, in revised form, April 10, 1985.