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THE CONSTRUCTION OF HARMONIC MAPS INTO COMPLEX GRASSMANNIANS

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To Professor J. Eells on his sixtieth birthday

Introduction

A. Background. In [17], following work of A. M. Din and W. J. Zakrzewski [9] and V. Glaser and R. Stora [20], J. Eells and the second author described, in terms of holomorphic maps, all harmonic maps (or, equivalently, minimal branched immersions) of the Riemann sphere S^2 to a complex projective space $\mathbf{C}P^n$ and all harmonic maps from a two-torus T^2 to $\mathbf{C}P^n$ of nonzero degree. (For the S^2 case see also D. Burns [3], and for a moving frames interpretation, S.-S. Chern and J. Wolfson [7], [32].) The harmonic maps were obtained by successive differentiations of a holomorphic map; this process gave all harmonic maps from any Riemannn surface to $\mathbf{C}P^n$ satisfying a certain "isotropy" property of orthogonality of iterated $(1,0)$ and $(0,1)$ derivatives. The vanishing of a sequence of holomorphic differentials (cf. [34]) then guaranteed isotropy in the S^2 and T^2 cases showing that all harmonic maps had been obtained.

Regarding $\mathbf{C}P^n$ as the complex Grassmannian $G_{1,n+1}$ of (complex) 1-planes in Euclidean $(n+1)$-space $\mathbf{C}^{n+1}$, it was natural to try to extend these results to give a description of all harmonic maps from S^2 to a complex Grassmannian in terms of "holomorphic data". In [19] S. Erdem and the second author showed how to construct all harmonic maps from any Riemann surface to a complex Grassmannian $G_{k,n}$ which satisfy a "strong isotropy" property, however, for $k > 1$, this did not give all harmonic maps from the Riemann sphere S^2 to $G_{k,n}$ (for further developments and related work see [10], [18], [21]). In [25] J. Ramanthan succeeded in describing all harmonic maps from the Riemann sphere to $G_{2,4}$ in terms of "holomorphic data" by which we shall henceforth mean holomorphic maps into a Grassmannian and holomorphic sections of fibre bundles over the domain. A. R. Aithal then dealt with the case

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