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TOTALLY GEODESIC FOLIATIONS ON 4-MANIFOLDS

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Abstract

We give a rather detailed description of the behavior of 2-dimensional totally geodesic foliations on compact Riemannian 4-manifolds. In particular, we obtain a complete characterization in the simply connected case.

0. Introduction

A foliation $\mathscr{F}$ is "geodesible" if there exists a Riemannian metric that makes $\mathscr{F}$ totally geodesic. The problem of characterizing geodesible foliations has been essentially solved in the 1-dimensional case [16]. There exists in fact numerous geodesible flows; for instance, every compact 3-manifold admits contact flows and such flows are geodesible [11]. Equally, the problem has been solved in the codimension one case [10]. Here the situation is so rigid that one can give a complete classification. The basic point is that, in codimension 1, the distribution $\mathscr{F}^{\perp}$, orthogonal to $\mathscr{F}$, is evidently integrable. In arbitrary codimension, there is a global description of the qualitative behavior of geodesible foliations that tries to take into account the nonintegrability of $\mathscr{F}^{\perp}$ [2], [4], [6]. However, this method cannot give rise to a complete classification. The first case where this analysis provides effective tools is that of 2-dimensional foliations on 4-manifolds. This is the object of study of this paper.

We assume that the foliation $\mathscr{F}$ and the manifold M are oriented and C^{∞} (for some general comments concerning the C^0 case, see [20]).

The following theorem has the advantage of splitting the problem into three subproblems. Note that, for the present, we treat 2-dimensional foliations of arbitrary codimension.

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