J. DIFFERENTIAL GEOMETRY
23 (1986) 207–240

THE WITTEN COMPLEX AND THE DEGENERATE MORSE INEQUALITIES

JEAN-MICHEL BISMUT

Abstract

In this paper, we use the complex introduced by Witten to prove the Morse inequalities, in the case where the critical points of the Morse function are isolated, and also the degenerate Morse inequalities of Bott. The method is based on a natural extension of the heat equation method for the proof of the index theorem. In the degenerate case, the de Rham complex is twisted using a nontrivial transformation of the exterior algebra on the neighborhood of the critical points. The cohomology of a manifold is compared to the L^2 cohomology of certain fiber bundles over the critical submanifolds. A L^2 version of the Thom isomorphism is then proved, from which the degenerate Morse inequalities follow.

0. Introduction

In a very interesting paper [19], Witten has shown how to prove analytically the Morse inequalities for a Morse function h with isolated critical points. The proof involves the construction of a family of complexes indexed by $t > 0$ associated with the operator $d^{h/t}$ defined by

$$d^{h/t} = e^{-h/t} d e^{h/t}. \tag{0.1}$$

By studying the lower part of the spectrum of the corresponding Laplacian $\square^{h/t}$ as $t \downarrow\downarrow 0$, Witten proved the Morse inequalities. Also, Witten suggested a method of proof of the degenerate Morse inequalities of Bott [9] when h has critical submanifolds.

In [7], we gave a probabilistic proof of the Atiyah-Singer index theorem and of the corresponding Lefschetz fixed point formulas, based on the heat

Received October 2, 1984. This work was partially supported by National Science Foundation Grant MCS-8108814(A02) while the author was visiting the Institute for Advanced Study, Princeton.