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AN INTEGRAL FORMULA FOR THE MEASURE OF RAYS ON COMPLETE OPEN SURFACES

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Dedicated to Professor Herbert Busemann on his 80th birthday

1. Introduction

It is interesting to investigate the influence of total curvature of a complete, noncompact, oriented and finitely connected Riemannian 2-manifold on the Riemannian metric. The total curvature of such an M is defined to be an improper integral of the Gaussian curvature G with respect to the area element dM induced from the Riemannian metric, and expressed as

$$c(M) = \int_M G\,dM.$$

The pioneering work on total curvature was done by Cohn-Vossen in [2], which states that if M admits total curvature, then $c(M) \leqslant 2\pi\chi(M)$, where $\chi(M)$ is the Euler characteristic of M. He also proved in [3] that if a Riemannian plane M (e.g., M is a complete Riemannian manifold homeomorphic to R^2) admits total curvature and if there exists a straight line (defined in the next paragraph) on M, then $c(M) \leqslant 0$.

It is the nature of completeness and noncompactness of a Riemannian plane M that through every point on M there passes at least a ray γ: $[0, \infty) \to M$, where it is a unit speed geodesic satisfying $d(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2|$ for all $t_1, t_2 > 0$, and d is the distance function induced from the Riemannian metric. A unit speed geodesic γ: $R \to M$ is called a straight line if $d(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2|$ for all $t_1, t_2 \in R$. Recall that M is said to be *finitely connected* if it is homeomorphic to a compact 2-manifold (without boundary) with finitely many points removed.

As was shown by Maeda [7], [8] and by Shiga [10], the total curvature of a Riemannian plane M imposes strong restrictions to the mass of rays emanating from an arbitrary fixed point on M. For a point p on M let S_p be the unit

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