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THE NORMALIZED CURVE SHORTENING FLOW AND HOMOTHETIC SOLUTIONS

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The curve shortening problem, by now widely known, is to understand the evolution of regular closed curves $\gamma\colon \mathbf{R}/\mathbf{Z} \to M$ moving according to the curvature normal vector: $\partial\gamma/\partial t = kN = -$"the L^2-gradient of arc length". One motivation for this problem has been the view expressed in this connection by C. Croke, H. Gluck, W. Ziller, and others: it would be desirable to improve on some complicated and ad hoc constructions that have been used in the theory of closed geodesics to iteratively shorten curves.

As a test case it has been a goal to prove the conjecture that kN generates a flow on the space of simple closed curves in the plane, preserving embeddedness and making such curves circular asymptotically as length approaches zero. However, the evolution equation for the curvature of γ_t turns out to be quite subtle, and the conjecture is not yet settled. Indeed, in the nonsimple case one generally expects singular behavior, and part of the intrinsic interest of the problem lies in the fact that the global condition of embeddedness is apparently recognized by the "near-sighted" equation.

What is known thus far is that the conjecture is true for convex curves, that simple curves do in fact remain simple (provided curvature stays bounded), and that short time solutions to the equations exist in full generality; these results are due to M. Gage and R. Hamilton (see [1], [2], [3]).

1. Main results

The starting point for the present investigation is a modification of the usual curve shortening flow; the flow is geometrically unchanged, but a tangential field bT is added to kN to maintain constant speed $\alpha = |\partial\gamma_t/\partial\sigma|$ along the curve.

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